A hierarchy of non-hydrostatic models for free-surface flows

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Joint work with
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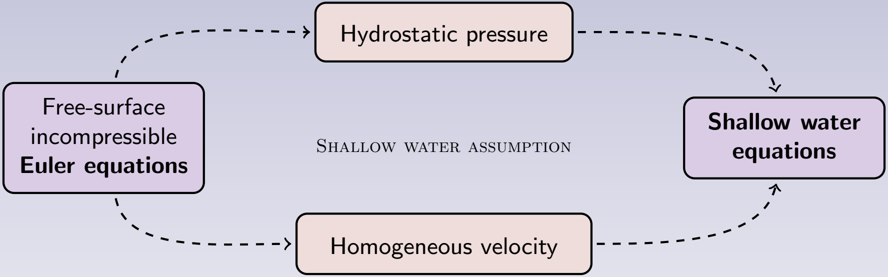
Outline

Literature about free-surface flows

Free-surface
incompressible
Euler equations

Literature about free-surface flows



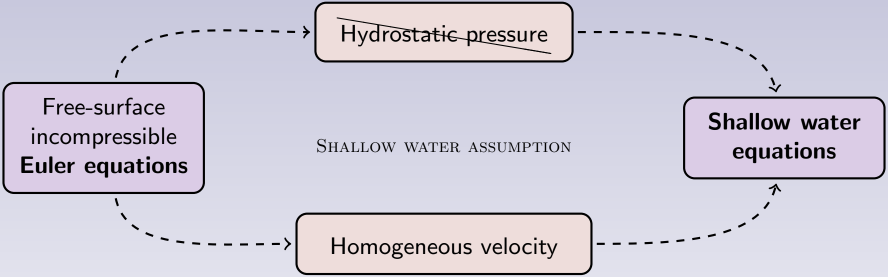
A. Barré de Saint-Venant, *Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leurs lits* (**C. R. Acad. Sci.** 73, 1871)

J.-F. Gerbeau, B. Perthame, *Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation* (**Discrete Contin. Dyn. Syst. Ser. B** 1(1), 2001)

S. Ferrari, F. Saleri, *A new two-dimensional Shallow Water model including pressure effects and slow varying bottom topography* (**Math. Model. Numer. Anal.** 38(2), 2004)

F. Marche, *Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects* (**Eur. J. Mech. B Fluids** 26(1), 2007)

Literature about free-surface flows



F. Serre, *Contribution à l'étude des écoulements permanents et variables dans les canaux* (**La Houille Blanche** 6, 1953)

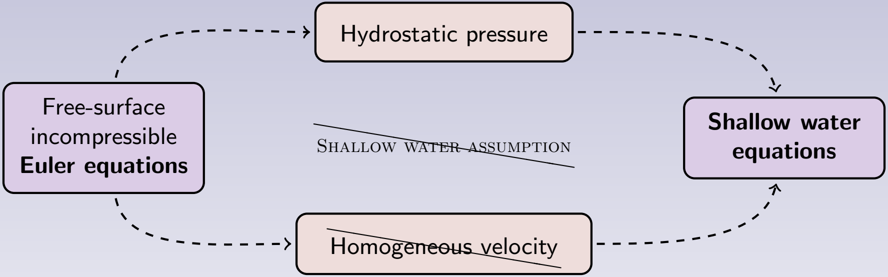
A.E. Green, P.M. Naghdi, *A derivation of equations for wave propagation in water of variable depth* (**J. Fluid Mech.** 78(2), 1976)

M.-O. Bristeau, J. Sainte-Marie, *Derivation of a non-hydrostatic shallow water model; Comparison with Saint-Venant and Boussinesq systems* (**Discrete Contin. Dyn. Syst. Ser. B** 10(4), 2008)

D. Lannes, P. Bonneton, *Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation* (**Phys. Fluids** 21(1), 2009)

Peregrine '67, Madsen *et al.* '91 '96 '03 '06, Nwogu '93, Casulli *et al.* '95 '99, Yamazaki *et al.* '09, ...

Literature about free-surface flows



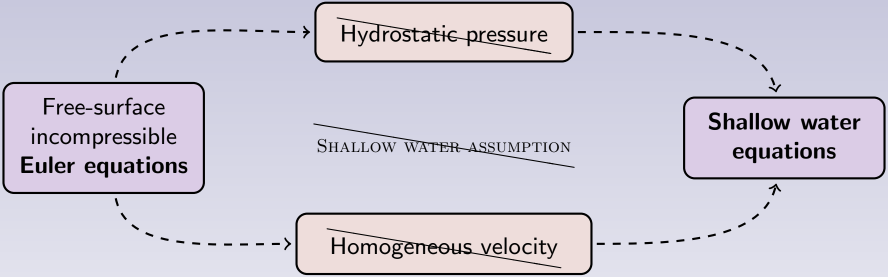
E. Audusse, M.-O. Bristeau, B. Perthame, J. Sainte-Marie, *A multilayer Saint-Venant system with mass exchanges for Shallow Water flows. Derivation and numerical validation* (**Math. Model. Numer. Anal.** 45(1), 2011)

F. Bouchut, V. Zeitlin, *A robust well-balanced scheme for multi-layer shallow water equations* (**Discrete Contin. Dyn. Syst. Ser. B** 13(4), 2010)

E.D. Fernández-Nieto, E.H. Koné, T. Morales de Luna, R. Bürger, *A multilayer shallow water system for polydisperse sedimentation* (**J. Comput. Phys.** 238, 2013)

Castro *et al.* '01 '04 '10, Narbona *et al.* '09 '13, Rambaud '11, . . .

Literature about free-surface flows

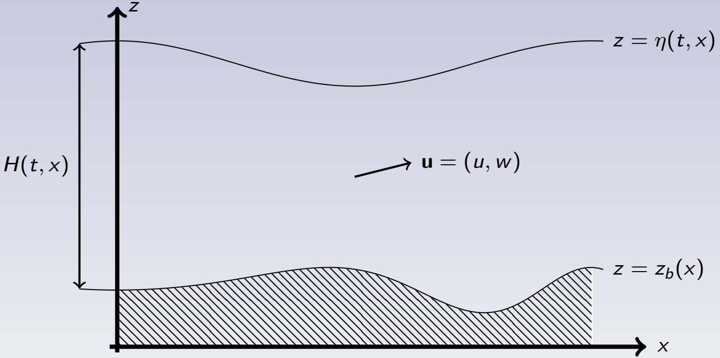


Derivation of multilayer non-hydrostatic models

M. Zijlema, G.S. Stelling, *Further experiences with computing non-hydrostatic free-surface flows involving water waves* (**Int. J. Numer. Methods Fluids** 48(2), 2005)

Y. Bai, K.F. Cheung, *Dispersion and nonlinearity of multi-layer non-hydrostatic free-surface flow* (**J. Fluid Mech.** 726, 2013)

Fluid domain



Water height:

$$H(t,x) = \eta(t,x) - z_b(x)$$

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2 + p) + \partial_z(uw) = 0 \\ \partial_t w + \partial_x(uw) + \partial_z(w^2 + p) = -g \end{cases}$$

set in the domain $\Omega(t) = \left\{(x, z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \leq z \leq \eta(t, x)\right\}$

Boundary conditions

$$\partial_t \eta(t, x) + u\big(t, x, \eta(t, x)\big)\partial_x \eta(t, x) - w\big(t, x, \eta(t, x)\big) = 0$$
$$p\big(t, x, \eta(t, x)\big) = p^{atm}(t, x)$$
$$u\big(t, x, z_b(x)\big)z_b'(x) - w\big(t, x, z_b(x)\big) = 0$$

together with well-prepared initial conditions

Pressure fields $p(t, x, z) = p^{atm}(t, x) + g\big(\eta(t, x) - z\big) + q(t, x, z)$

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2 + q) + \partial_z(uw) = -\partial_x(g\eta + p^{atm}) \\ \partial_t w + \partial_x(uw) + \partial_z(w^2 + q) = 0 \end{cases}$$

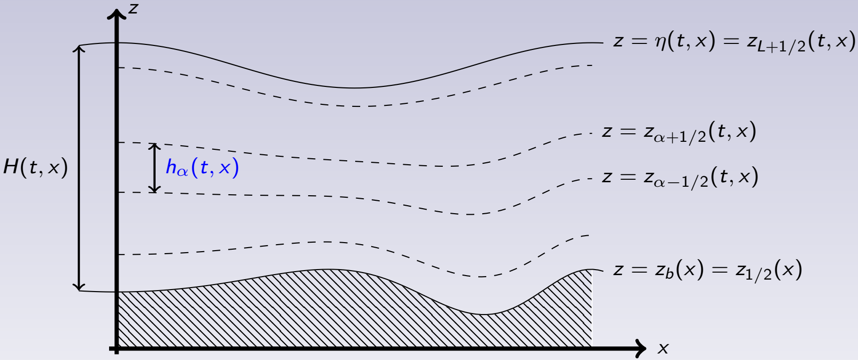
set in the domain $\Omega(t) = \left\{(x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \le z \le \eta(t,x)\right\}$

Boundary conditions

$$\partial_t \eta(t,x) + u\big(t,x,\eta(t,x)\big)\partial_x \eta(t,x) - w\big(t,x,\eta(t,x)\big) = 0$$
$$q\big(t,x,\eta(t,x)\big) = 0$$
$$u\big(t,x,z_b(x)\big)z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

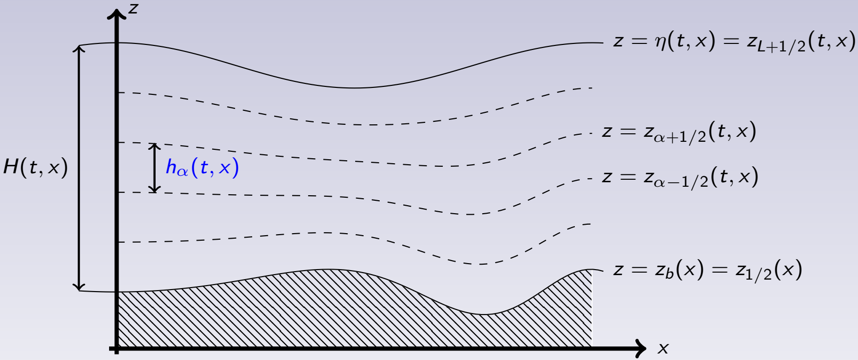
together with well-prepared initial conditions

Multilayer framework



Height decomposition: $h_\alpha(t,x) = \ell_\alpha H(t,x)$ with $\ell_\alpha \in (0,1)$ and $\sum_{\alpha=1}^{L} \ell_\alpha = 1$

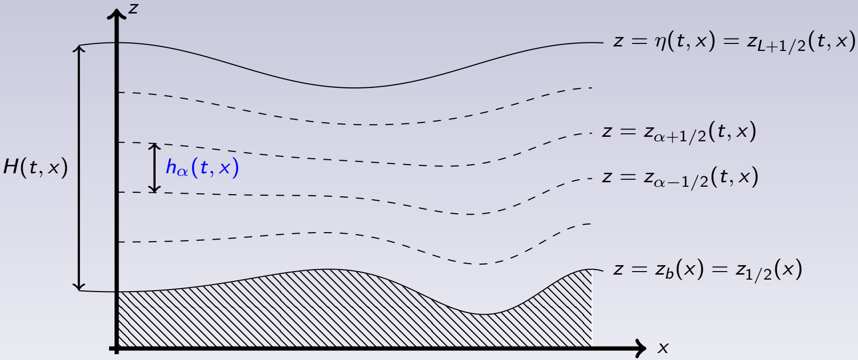
Multilayer framework



Height decomposition: $h_\alpha(t,x) = \ell_\alpha H(t,x)$ with $\ell_\alpha \in (0,1)$ and $\sum_{\alpha=1}^{L} \ell_\alpha = 1$

Homogeneous mesh: $\ell_\alpha = \dfrac{1}{L}$

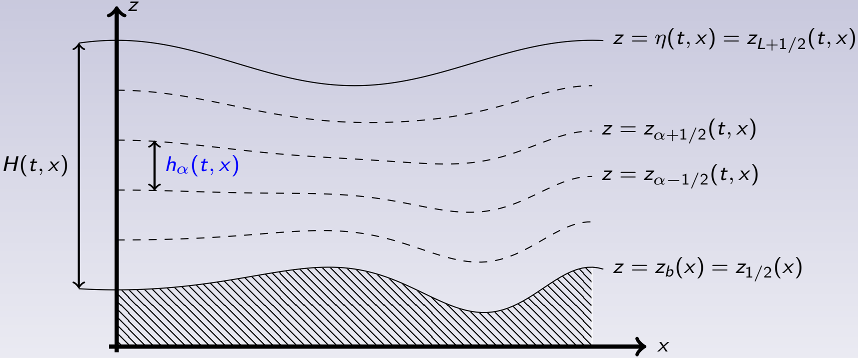
Multilayer framework



Notations

$$[\![f]\!]_{\alpha+1/2} = f^+_{\alpha+1/2} - f^-_{\alpha+1/2}, \quad \widetilde{f}_{\alpha+1/2} = \gamma_{\alpha+1/2} f^-_{\alpha+1/2} + (1-\gamma_{\alpha+1/2}) f^+_{\alpha+1/2}$$

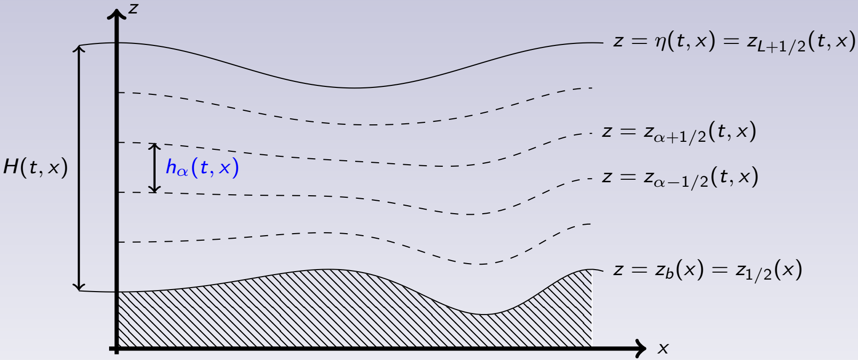
Multilayer framework



Notations

$$\langle f \rangle_\alpha(t,x) = \frac{1}{h_\alpha(t,x)} \int_{z_{\alpha-1/2}(t,x)}^{z_{\alpha+1/2}(t,x)} f(t,x,z)\,\mathrm{d}z$$

Multilayer framework



Notations

$$\mathbf{n}_{\alpha+1/2} = \left(-\partial_x z_{\alpha+1/2}, 1\right)^T$$

Discontinuous Galerkin framework

Let us be given a velocity field satisfying

$$[\![\mathbf{u}]\!]_{\alpha+1/2} \cdot \mathbf{n}_{\alpha+1/2} = 0 \iff [\![w]\!]_{\alpha+1/2} = [\![u]\!]_{\alpha+1/2} \partial_x z_{\alpha+1/2}$$

Toy model

$$\partial_t \mathscr{R} + \partial_x (u\mathscr{R} + \mathscr{P}) + \partial_z (w\mathscr{R} + \mathscr{Q}) = \mathscr{S} \tag{1}$$

where $\mathscr{R}$, $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{S}$ take values in $\mathbb{R}^p$

Semi-discrete formulation over each layer $\mathcal{L}_\alpha = (z_{\alpha+1/2}, z_{\alpha-1/2})$

$$\partial_t (h_\alpha \overline{\mathscr{R}}_\alpha) + \partial_x (h_\alpha [\overline{u\mathscr{R}}_\alpha + \overline{\mathscr{P}}_\alpha]) + \mathscr{F}^{\mathscr{R}}_{\alpha+1/2} - \mathscr{F}^{\mathscr{R}}_{\alpha-1/2} = h_\alpha \overline{\mathscr{S}}_\alpha$$

where

$$\mathscr{F}^{\mathscr{R}}_{\alpha+1/2} = \Upsilon_{\alpha+1/2} \widetilde{\mathscr{R}}_{\alpha+1/2} - \widetilde{\mathscr{P}}_{\alpha+1/2} \partial_x z_{\alpha+1/2} + \widetilde{\mathscr{Q}}_{\alpha+1/2}$$
$$\Upsilon_{\alpha+1/2} = \widetilde{w}_{\alpha+1/2} - \partial_t z_{\alpha+1/2} - \widetilde{u}_{\alpha+1/2} \partial_x z_{\alpha+1/2}$$

Discontinuous Galerkin framework

Let us be given a velocity field satisfying

$$[\![\mathbf{u}]\!]_{\alpha+1/2}\cdot\mathbf{n}_{\alpha+1/2}=0 \iff [\![w]\!]_{\alpha+1/2}=[\![u]\!]_{\alpha+1/2}\partial_x z_{\alpha+1/2}$$

Toy model

$$\partial_t\mathscr{R}+\partial_x(u\mathscr{R}+\mathscr{P})+\partial_z(w\mathscr{R}+\mathscr{Q})=\mathscr{S} \tag{1}$$

where $\mathscr{R}$, $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{S}$ take values in $\mathbb{R}^p$

Semi-discrete formulation over each layer $\mathcal{L}_\alpha=(z_{\alpha+1/2},z_{\alpha-1/2})$

$$\partial_t(h_\alpha\overline{\mathscr{R}}_\alpha)+\partial_x(h_\alpha[\overline{u\mathscr{R}}_\alpha+\overline{\mathscr{P}}_\alpha])+\mathscr{F}^{\mathscr{R}}_{\alpha+1/2}-\mathscr{F}^{\mathscr{R}}_{\alpha-1/2}=h_\alpha\overline{\mathscr{S}}_\alpha$$

where

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$$\Upsilon_{\alpha+1/2}=w^-_{\alpha+1/2}-\partial_t z_{\alpha+1/2}-u^-_{\alpha+1/2}\partial_x z_{\alpha+1/2}$$

Discontinuous Galerkin framework

Let us be given a velocity field satisfying

$$[\![\mathbf{u}]\!]_{\alpha+1/2} \cdot \mathbf{n}_{\alpha+1/2} = 0 \iff [\![w]\!]_{\alpha+1/2} = [\![u]\!]_{\alpha+1/2} \partial_x z_{\alpha+1/2}$$

Toy model

$$\partial_t \mathscr{R} + \partial_x (u\mathscr{R} + \mathscr{P}) + \partial_z (w\mathscr{R} + \mathscr{Q}) = \mathscr{S} \tag{1}$$

where $\mathscr{R}$, $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{S}$ take values in $\mathbb{R}^p$

Semi-discrete formulation over each layer $\mathcal{L}_\alpha = (z_{\alpha+1/2}, z_{\alpha-1/2})$

$$\partial_t (h_\alpha \overline{\mathscr{R}}_\alpha) + \partial_x (h_\alpha [\overline{u\mathscr{R}}_\alpha + \overline{\mathscr{P}}_\alpha]) + \mathscr{F}^{\mathscr{R}}_{\alpha+1/2} - \mathscr{F}^{\mathscr{R}}_{\alpha-1/2} = h_\alpha \overline{\mathscr{S}}_\alpha$$

where

$$\mathscr{F}^{\mathscr{R}}_{\alpha+1/2} = \Upsilon_{\alpha+1/2} \widetilde{\mathscr{R}}_{\alpha+1/2} - \widetilde{\mathscr{P}}_{\alpha+1/2} \partial_x z_{\alpha+1/2} + \widetilde{\mathscr{Q}}_{\alpha+1/2}$$
$$\Upsilon_{\alpha+1/2} = w^+_{\alpha+1/2} - \partial_t z_{\alpha+1/2} - u^+_{\alpha+1/2} \partial_x z_{\alpha+1/2}$$

Main idea

Discontinuities across layer interfaces allowed provided jump conditions

$$\partial_t \mathcal{Z}[\![\mathscr{R}]\!]_{z=\mathcal{Z}} + \partial_x \mathcal{Z}[\![u\mathscr{R} + \mathscr{P}]\!]_{z=\mathcal{Z}} - [\![w\mathscr{R} + \mathscr{Q}]\!]_{z=\mathcal{Z}} = 0$$

or equivalently

$$\Upsilon[\![\mathscr{R}]\!] - \partial_x \mathcal{Z}[\![\mathscr{P}]\!] + [\![\mathscr{Q}]\!] = 0$$

Integrating Eq. (1) over a layer $\mathcal{L}_\alpha$ yields

$$\begin{aligned} h_\alpha \langle \mathscr{S} \rangle_\alpha = {} & \partial_t (h_\alpha \langle \mathscr{R} \rangle_\alpha) - \mathscr{R}^-_{\alpha+1/2} \partial_t z_{\alpha+1/2} + \mathscr{R}^+_{\alpha-1/2} \partial_t z_{\alpha-1/2} \\ & + \partial_x (h_\alpha \langle u\mathscr{R} + \mathscr{P} \rangle_\alpha) - (u^-_{\alpha+1/2} \mathscr{R}^-_{\alpha+1/2} + \mathscr{P}^-_{\alpha+1/2}) \partial_x z_{\alpha+1/2} \\ & + (u^+_{\alpha-1/2} \mathscr{R}^+_{\alpha-1/2} + \mathscr{P}^+_{\alpha-1/2}) \partial_x z_{\alpha-1/2} \\ & + w^-_{\alpha+1/2} \mathscr{R}^-_{\alpha+1/2} + \mathscr{Q}^-_{\alpha+1/2} - w^+_{\alpha-1/2} \mathscr{R}^+_{\alpha-1/2} + \mathscr{Q}^+_{\alpha-1/2} \end{aligned}$$

Main idea

Discontinuities across layer interfaces allowed provided jump conditions

$$\partial_t \mathcal{Z}[\![\mathscr{R}]\!]_{z=\mathcal{Z}} + \partial_x \mathcal{Z}[\![u\mathscr{R} + \mathscr{P}]\!]_{z=\mathcal{Z}} - [\![w\mathscr{R} + \mathscr{Q}]\!]_{z=\mathcal{Z}} = 0$$

or equivalently

$$\Upsilon[\![\mathscr{R}]\!] - \partial_x \mathcal{Z}[\![\mathscr{P}]\!] + [\![\mathscr{Q}]\!] = 0$$

Integrating Eq. (1) over a layer $\mathcal{L}_\alpha$ yields

$$\begin{aligned} h_\alpha \langle \mathscr{S} \rangle_\alpha = \partial_t (h_\alpha \langle \mathscr{R} \rangle_\alpha) + \partial_x (h_\alpha \langle u\mathscr{R} + \mathscr{P} \rangle_\alpha) \\ + \left[\mathscr{R}^-_{\alpha+1/2} \Upsilon_{\alpha+1/2} - \mathscr{P}^-_{\alpha+1/2} \partial_x z_{\alpha+1/2} + \mathscr{Q}^-_{\alpha+1/2} \right] \\ - \left[\mathscr{R}^+_{\alpha-1/2} \Upsilon_{\alpha-1/2} - \mathscr{P}^+_{\alpha-1/2} \partial_x z_{\alpha-1/2} + \mathscr{Q}^+_{\alpha-1/2} \right] \end{aligned}$$

Main idea

Discontinuities across layer interfaces allowed provided jump conditions

$$\partial_t \mathcal{Z} [\![\mathscr{R}]\!]_{z=\mathcal{Z}} + \partial_x \mathcal{Z} [\![u\mathscr{R} + \mathscr{P}]\!]_{z=\mathcal{Z}} - [\![w\mathscr{R} + \mathscr{Q}]\!]_{z=\mathcal{Z}} = 0$$

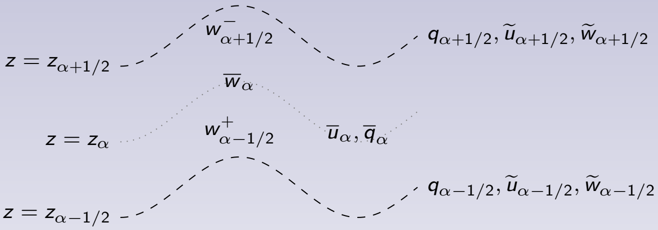
or equivalently

$$\Upsilon [\![\mathscr{R}]\!] - \partial_x \mathcal{Z} [\![\mathscr{P}]\!] + [\![\mathscr{Q}]\!] = 0$$

Integrating Eq. (1) over a layer $\mathcal{L}_\alpha$ yields

$$\begin{aligned} h_\alpha \langle \mathscr{S} \rangle_\alpha = \partial_t (h_\alpha \langle \mathscr{R} \rangle_\alpha) + \partial_x (h_\alpha \langle u\mathscr{R} + \mathscr{P} \rangle_\alpha) \\ + \left[\widetilde{\mathscr{R}}_{\alpha+1/2} \Upsilon_{\alpha+1/2} - \widetilde{\mathscr{P}}_{\alpha+1/2} \partial_x z_{\alpha+1/2} + \widetilde{\mathscr{Q}}_{\alpha+1/2} \right] \\ - \left[\widetilde{\mathscr{R}}_{\alpha-1/2} \Upsilon_{\alpha-1/2} - \widetilde{\mathscr{P}}_{\alpha-1/2} \partial_x z_{\alpha-1/2} + \widetilde{\mathscr{Q}}_{\alpha-1/2} \right] \end{aligned}$$

Spaces of approximation



$$u(t,x) = \sum_{\alpha=1}^{L} \overline{u}_\alpha(t,x) \mathbb{1}_{\{\mathcal{L}_\alpha(t,x)\}}(z) + \mathcal{E}_L$$

$$w(t,x) = \sum_{\alpha=1}^{L} \left[\overline{w}_\alpha(t,x) - \big(z - z_\alpha(t,x)\big)\partial_x \overline{u}_\alpha(t,x)\right] \mathbb{1}_{\{\mathcal{L}_\alpha(t,x)\}}(z) + \mathcal{E}'_L$$

q continuous over the water column

Core of the models

Applying the previous semi-discretisation to the Euler equations leads to

$$\begin{cases} \partial_t h_\alpha + \partial_x(h_\alpha \overline{u}_\alpha) + \Upsilon_{\alpha+1/2} - \Upsilon_{\alpha-1/2} = 0 \\ \partial_t(h_\alpha \overline{u}_\alpha) + \partial_x\left(h_\alpha \overline{u}_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm}) \\ \partial_t(h_\alpha \overline{w}_\alpha) + \partial_x\left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha\right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0 \end{cases}$$

with

$$\begin{cases} \mathcal{U}_{\alpha+1/2} = \widetilde{u}_{\alpha+1/2}\Upsilon_{\alpha+1/2} - \partial_x z_{\alpha+1/2} q_{\alpha+1/2} \\ \mathcal{W}_{\alpha+1/2} = \widetilde{w}_{\alpha+1/2}\Upsilon_{\alpha+1/2} + q_{\alpha+1/2} \end{cases}$$

and

$$\Upsilon_{\alpha+1/2} = \widetilde{w}_{\alpha+1/2} - \partial_t z_{\alpha+1/2} - \widetilde{u}_{\alpha+1/2}\partial_x z_{\alpha+1/2}$$

Core of the models

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$$\begin{cases} \partial_t h_\alpha + \partial_x(h_\alpha \overline{u}_\alpha) + \Upsilon_{\alpha+1/2} - \Upsilon_{\alpha-1/2} = 0 \\ \partial_t(h_\alpha \overline{u}_\alpha) + \partial_x\left(h_\alpha \overline{u}_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm}) \\ \partial_t(h_\alpha \overline{w}_\alpha) + \partial_x\left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha\right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0 \end{cases}$$

with

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and

$$\Upsilon_{\alpha+1/2} = \sum_{\beta=\alpha+1}^{L} \partial_x\left(h_\beta\left[\overline{u}_\beta - \overline{\overline{u}}\right]\right) \quad \text{where} \quad \overline{\overline{u}} = \sum_{\alpha=1}^{L} \ell_\alpha \overline{u}_\alpha$$

Core of the models

Applying the previous semi-discretisation to the Euler equations leads to

$$\begin{cases} \partial_t H + \partial_x \left(H\overline{\overline{u}} \right) = 0 \\ \partial_t (h_\alpha \overline{u}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + h_\alpha \overline{q}_\alpha \right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x (g\eta + p^{atm}) \\ \partial_t (h_\alpha \overline{w}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha \right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0 \end{cases}$$

with

$$\begin{cases} \mathcal{U}_{\alpha+1/2} = \widetilde{u}_{\alpha+1/2} \Upsilon_{\alpha+1/2} - \partial_x z_{\alpha+1/2} q_{\alpha+1/2} \\ \mathcal{W}_{\alpha+1/2} = \widetilde{w}_{\alpha+1/2} \Upsilon_{\alpha+1/2} + q_{\alpha+1/2} \end{cases}$$

and

$$\Upsilon_{\alpha+1/2} = \sum_{\beta=\alpha+1}^{L} \partial_x \left(h_\beta \left[\overline{u}_\beta - \overline{\overline{u}} \right] \right) \quad \text{where} \quad \overline{\overline{u}} = \sum_{\alpha=1}^{L} \ell_\alpha \overline{u}_\alpha$$

Requirements for the $\mathbb{P}_1$ choice

Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t\left(h_\alpha \langle zw\rangle_\alpha\right) + \partial_x\left(h_\alpha \overline{u}_\alpha \langle zw\rangle_\alpha\right) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha \left\langle w^2 + q\right\rangle_\alpha$$

Requirements for the $\mathbb{P}_1$ choice ($\mathcal{S}_1$ model)

Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t(h_\alpha\langle zw\rangle_\alpha) + \partial_x(h_\alpha\overline{u}_\alpha\langle zw\rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha\big\langle w^2+q\big\rangle_\alpha$$

Let us introduce the signed standard deviation $\sigma_\alpha = -\frac{h_\alpha\partial_x\overline{u}_\alpha}{2\sqrt{3}}$ such that

$$\big\langle w^2\big\rangle_\alpha = \overline{w}_\alpha^2 + \sigma_\alpha^2, \qquad \langle zw\rangle_\alpha = z_\alpha\overline{w}_\alpha + \frac{h_\alpha\sigma_\alpha}{2\sqrt{3}}$$

Requirements for the $\mathbb{P}_1$ choice ($\mathcal{S}_1$ model)

Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

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$$\partial_t(h_\alpha \langle zw\rangle_\alpha) + \partial_x(h_\alpha \overline{u}_\alpha \langle zw\rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha \left\langle w^2 + q\right\rangle_\alpha$$

Then

$$\begin{aligned}\partial_t(h_\alpha z_\alpha \overline{w}_\alpha) + \partial_x(h_\alpha z_\alpha \overline{u}_\alpha \overline{w}_\alpha) + \partial_t\left(\frac{h_\alpha^2\sigma_\alpha}{2\sqrt{3}}\right) + \partial_x\left(\frac{h_\alpha^2\sigma_\alpha\overline{u}_\alpha}{2\sqrt{3}}\right)\\ + z_{\alpha+1/2}(\widetilde{w}_{\alpha+1/2}\Upsilon_{\alpha+1/2} + q_{\alpha+1/2}) - z_{\alpha-1/2}(\widetilde{w}_{\alpha-1/2}\Upsilon_{\alpha-1/2} + q_{\alpha-1/2})\\ = h_\alpha\left(\overline{w}_\alpha^2 + \sigma_\alpha^2 + \overline{q}_\alpha\right)\end{aligned}$$

Requirements for the $\mathbb{P}_1$ choice ($\mathcal{S}_1$ model)

Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t(h_\alpha\langle zw\rangle_\alpha) + \partial_x(h_\alpha\overline{u}_\alpha\langle zw\rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha\big\langle w^2+q\big\rangle_\alpha$$

Then

$$\begin{aligned}\partial_t(h_\alpha\sigma_\alpha) + \partial_x(h_\alpha\sigma_\alpha\overline{u}_\alpha) = 2\sqrt{3}\,\Bigg[\overline{q}_\alpha - \frac{q_{\alpha+1/2}+q_{\alpha-1/2}}{2}\\ - \Upsilon_{\alpha+1/2}\left(\frac{h_\alpha\partial_x\overline{u}_\alpha}{12} + \frac{\widetilde{w}_{\alpha+1/2}-\overline{w}_\alpha}{2}\right)\\ +\Upsilon_{\alpha-1/2}\left(\frac{h_\alpha\partial_x\overline{u}_\alpha}{12} + \frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{2}\right)\Bigg]\end{aligned}$$

Requirements for the $\mathbb{P}_1$ choice ($\mathcal{S}_{1/2}$ model)

Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t(h_\alpha \langle zw\rangle_\alpha) + \partial_x(h_\alpha \overline{u}_\alpha \langle zw\rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha \big\langle w^2 + q\big\rangle_\alpha$$

Rather using a Hermitte interpolation leads to

$$zw_{|\mathcal{L}_\alpha} \approx z_\alpha \overline{w}_\alpha + (z - z_\alpha)(\overline{w}_\alpha - z_\alpha \partial_x \overline{u}_\alpha), \qquad w^2_{|\mathcal{L}_\alpha} \approx \overline{w}^2_\alpha - 2(z - z_\alpha)\overline{w}_\alpha \partial_x \overline{u}_\alpha$$

Requirements for the $\mathbb{P}_1$ choice ($\mathcal{S}_{1/2}$ model)

Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t(h_\alpha \langle zw\rangle_\alpha) + \partial_x(h_\alpha \overline{u}_\alpha \langle zw\rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha \big\langle w^2 + q\big\rangle_\alpha$$

Then

$$\begin{aligned}\partial_t(h_\alpha z_\alpha \overline{w}_\alpha) + \partial_x(h_\alpha z_\alpha \overline{u}_\alpha \overline{w}_\alpha) + z_{\alpha+1/2}q_{\alpha+1/2} - z_{\alpha-1/2}q_{\alpha-1/2} \\ + \Upsilon_{\alpha+1/2}\left(z_{\alpha+1/2}\widetilde{w}_{\alpha+1/2} + \frac{H^2}{4L^2}\widetilde{(\partial_x \overline{u})}_{\alpha+1/2}\right) \\ - \Upsilon_{\alpha-1/2}\left(z_{\alpha-1/2}\widetilde{w}_{\alpha-1/2} + \frac{H^2}{4L^2}\widetilde{(\partial_x \overline{u})}_{\alpha-1/2}\right) = h_\alpha\left(\overline{w}^2_\alpha + \overline{q}_\alpha\right)\end{aligned}$$

Requirements for the $\mathbb{P}_1$ choice ($\mathcal{S}_{1/2}$ model)

Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t(h_\alpha \langle zw\rangle_\alpha) + \partial_x(h_\alpha \overline{u}_\alpha \langle zw\rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha \big\langle w^2 + q\big\rangle_\alpha$$

Then

$$\overline{q}_\alpha = \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} + \Upsilon_{\alpha+1/2}\left(\frac{H}{4L}\widetilde{(\partial_x \overline{u})}_{\alpha+1/2} + \frac{\widetilde{w}_{\alpha+1/2} - \overline{w}_\alpha}{2}\right) - \Upsilon_{\alpha-1/2}\left(\frac{H}{4L}\widetilde{(\partial_x \overline{u})}_{\alpha-1/2} + \frac{\overline{w}_\alpha - \widetilde{w}_{\alpha-1/2}}{2}\right)$$

$\mathcal{S}_1$ model

$$\partial_t H + \partial_x \left(H\overline{\overline{u}}\right) = 0$$

$$\partial_t(h_\alpha \overline{u}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x (g\eta + p^{atm})$$

$$\partial_t(h_\alpha \overline{w}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha\right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\partial_t(h_\alpha \sigma_\alpha) + \partial_x(h_\alpha \sigma_\alpha \overline{u}_\alpha) = 2\sqrt{3}\left[\overline{q}_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} - \Upsilon_{\alpha+1/2}\left(\frac{h_\alpha \partial_x \overline{u}_\alpha}{12} + \frac{\widetilde{w}_{\alpha+1/2} - \overline{w}_\alpha}{2}\right) + \Upsilon_{\alpha-1/2}\left(\frac{h_\alpha \partial_x \overline{u}_\alpha}{12} + \frac{\overline{w}_\alpha - \widetilde{w}_{\alpha-1/2}}{2}\right)\right]$$

$$\partial_x \overline{u}_\alpha + \frac{w^-_{\alpha+1/2} - \overline{w}_\alpha}{h_\alpha/2} = 0 \qquad \sigma_\alpha = -\frac{h_\alpha \partial_x \overline{u}_\alpha}{2\sqrt{3}}$$

$$w^+_{\alpha-1/2} - \overline{u}_\alpha \partial_x z_{\alpha-1/2} + \sum_{\beta=1}^{\alpha-1} \partial_x (h_\beta \overline{u}_\beta) = 0$$

$\mathcal{S}_{1/2}$ model

$$\partial_t H + \partial_x \left(H\overline{\overline{u}}\right) = 0$$

$$\partial_t(h_\alpha \overline{u}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm})$$

$$\partial_t(h_\alpha \overline{w}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha\right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\overline{q}_\alpha = \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} + \Upsilon_{\alpha+1/2}\left(\frac{H}{4L}\widetilde{(\partial_x \overline{u})}_{\alpha+1/2} + \frac{\widetilde{w}_{\alpha+1/2} - \overline{w}_\alpha}{2}\right) - \Upsilon_{\alpha-1/2}\left(\frac{H}{4L}\widetilde{(\partial_x \overline{u})}_{\alpha-1/2} + \frac{\overline{w}_\alpha - \widetilde{w}_{\alpha-1/2}}{2}\right)$$

$$\partial_x \overline{u}_\alpha + \frac{w^-_{\alpha+1/2} - \overline{w}_\alpha}{h_\alpha/2} = 0 \qquad \sigma_\alpha = -\frac{h_\alpha \partial_x \overline{u}_\alpha}{2\sqrt{3}}$$

$$w^+_{\alpha-1/2} - \overline{u}_\alpha \partial_x z_{\alpha-1/2} + \sum_{\beta=1}^{\alpha-1} \partial_x(h_\beta \overline{u}_\beta) = 0$$

$\mathcal{S}_0$ model

$$\partial_t H + \partial_x \left(H\overline{\overline{u}} \right) = 0$$

$$\partial_t(h_\alpha \overline{u}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + h_\alpha \overline{q}_\alpha \right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x (g\eta + p^{atm})$$

$$\partial_t(h_\alpha \overline{w}_\alpha) + \partial_x \left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha \right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\overline{q}_\alpha = \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2}$$

$$\overline{w}_\alpha - \overline{u}_\alpha \partial_x z_\alpha + \sum_{\beta=1}^{\alpha-1} \partial_x (h_\beta \overline{u}_\beta) + \frac{1}{2} \partial_x (h_\alpha \overline{u}_\alpha) = 0$$

Energy inequality

Denoting $\mathcal{K} = \frac{u^2+w^2}{2}$, smooth solutions to the Euler equations satisfy

$$\partial_t \left(\int_{z_b}^{\eta} \left(\mathcal{K} + g\frac{\eta + z_b}{2} + p^{atm} \right) \mathrm{d}z \right) + \partial_x \left(\int_{z_b}^{\eta} u(\mathcal{K} + q + g\eta + p^{atm}) \, \mathrm{d}z \right) = H\partial_t p^{atm}.$$

Proposition

Let us assume that $\left(\gamma_{\alpha+1/2} - \frac{1}{2}\right) \Upsilon_{\alpha+1/2} \geq 0$*. If* $(H, \overline{u}_\alpha, \overline{w}_\alpha, \overline{q}_\alpha)$ *are smooth solutions to the* $\mathcal{S}_1$*-model, then with* $\overline{\mathcal{K}}_\alpha = \frac{\overline{u}_\alpha^2 + \overline{w}_\alpha^2 + \sigma_\alpha^2}{2}$

$$\partial_t \left[\sum_{\alpha=1}^{L} h_\alpha \left(\overline{\mathcal{K}}_\alpha + g z_\alpha + p^{atm} \right) \right] + \partial_x \left[\sum_{\alpha=1}^{L} h_\alpha \overline{u}_\alpha \left(\overline{\mathcal{K}}_\alpha + \overline{q}_\alpha + g\eta + p^{atm} \right) \right] \leq H\partial_t p^{atm}.$$

Comment

Constraint $\left(\gamma_{\alpha+1/2} - \frac{1}{2}\right)\Upsilon_{\alpha+1/2} \geq 0$ is equivalent to taking

$$\gamma_{\alpha+1/2} = \frac{1}{2}\left(1 + \lambda\,\mathrm{sign}(\Upsilon_{\alpha+1/2})\right)$$

for any $\lambda \geq 0$, which gives

$$\widetilde{\mathscr{R}}_{\alpha+1/2}\Upsilon_{\alpha+1/2} = \frac{\mathscr{R}^+_{\alpha+1/2} + \mathscr{R}^-_{\alpha+1/2}}{2}\Upsilon_{\alpha+1/2} - \frac{\lambda}{2}|\Upsilon_{\alpha+1/2}|\left(\mathscr{R}^+_{\alpha+1/2} - \mathscr{R}^-_{\alpha+1/2}\right).$$

The energy inequality is satisfied in particular for $\gamma_{\alpha+1/2} = \frac{1}{2}$ ($\lambda = 0$) and for $\gamma_{\alpha+1/2} = \mathbb{1}_{\{\Upsilon_{\alpha+1/2}\geq 0\}}$ ($\lambda = 1$).

Hydrodynamic balances

Proposition

Let $(H, \overline{u}_\alpha, \overline{w}_\alpha, q_{\alpha+1/2})$ *be smooth solutions to the* $\mathcal{S}_1$ *model. Then*

- *The conservation of global volume:* $\partial_t \left(\int_{\mathbb{R}} H(t,x)\,\mathrm{d}x \right) = 0$
- *The balance of horizontal momentum:*

$$\partial_t \left(\int_{\mathbb{R}} H(t,x)\overline{\overline{u}}(t,x)\,\mathrm{d}x \right) = -\int_{\mathbb{R}} \left[H(t,x)\partial_x p^{atm}(t,x) + \big(gH(t,x) + q_{1/2}(t,x)\big)\partial_x z_b(x) \right] \mathrm{d}x$$

- *The balance of vertical momentum:*

$$\partial_t \left(\int_{\mathbb{R}} H(t,x)\overline{\overline{w}}(t,x)\,\mathrm{d}x \right) = -\int_{\mathbb{R}} q_{1/2}(t,x)\,\mathrm{d}x$$

Dispersion relations

Let us linearise around the so-called lake-at-rest steady state $(H_0, 0, 0, 0)$.

Proposition

There exists a plane wave solution $\left(\hat{H}, \hat{u}_\alpha, \hat{w}_\alpha, \hat{q}_\alpha\right) e^{i(kx-\omega t)}$ *to the linearised* $\mathcal{S}_1$ *system provided the following dispersion relation holds*

$$\omega^2 = k^2 c_{sw}^2 \left\langle \mathcal{A}_{kH_0}^{-1} \mathbf{e}, \boldsymbol{\ell} \right\rangle$$

where $c_{sw} = \sqrt{gH_0}$, $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_L) \in \mathbb{R}^L$, $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^L$ *and*

$$\mathcal{A}_x = \mathcal{I}_L + x^2 \mathcal{B}, \quad \text{with} \quad \mathcal{B}_{\alpha\beta} = -\frac{\ell_\alpha^2}{6} \delta_{\alpha\beta} + \ell_\beta \left[\frac{\ell_{\max\{\alpha,\beta\}}}{2} + \sum_{\gamma=\max\{\alpha,\beta\}+1}^{L} \ell_\gamma \right]$$

Dispersion relations

Let us linearise around the so-called lake-at-rest steady state $(H_0, 0, 0, 0)$.

Proposition

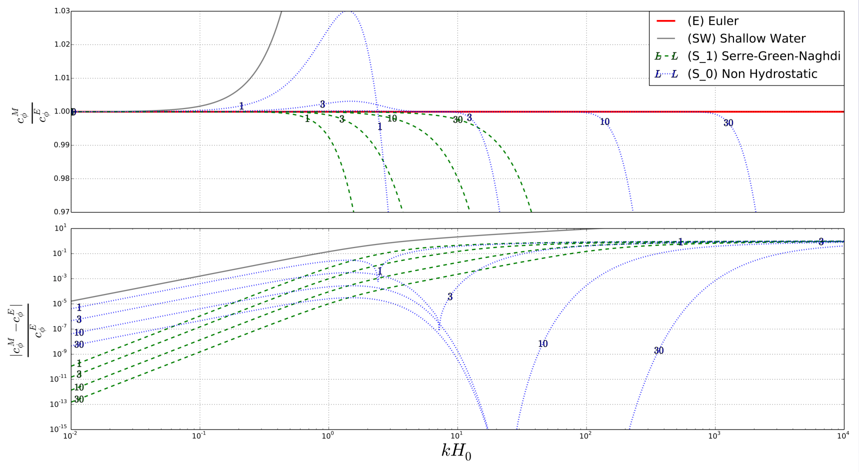
There exists a plane wave solution $\left(\hat{H}, \hat{u}_\alpha, \hat{w}_\alpha, \hat{q}_\alpha\right) e^{i(kx-\omega t)}$ *to the linearised* $\mathcal{S}_0$ *system provided the following dispersion relation holds*

$$\omega^2 = k^2 c_{sw}^2 \left\langle \mathcal{A}_{kH_0}^{-1} \mathbf{e}, \boldsymbol{\ell} \right\rangle$$

where $c_{sw} = \sqrt{gH_0}$, $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_L) \in \mathbb{R}^L$, $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^L$ *and*

$$\mathcal{A}_x = \mathcal{I}_L + x^2 \mathcal{B}, \quad \text{with} \quad \mathcal{B}_{\alpha\beta} = -\frac{\ell_\alpha^2}{4}\delta_{\alpha\beta} + \ell_\beta \left[\frac{\ell_{\max\{\alpha,\beta\}}}{2} + \sum_{\gamma=\max\{\alpha,\beta\}+1}^{L} \ell_\gamma \right]$$

Phase velocity



Conclusion

Summary

- Derivation of a class of multilayer non-hydrostatic models as semi-discretisations of the Euler equations
- Proof of physical properties (energy, hydrodynamic balances, dispersive effects)

On-going works

- Convergence of the dispersion relation
- Numerical simulations
- Incorporation of viscous effects
- Enriching the physics

E. Fernández-Nieto, M. Parisot, Y. Penel & J. Sainte-Marie, *Layer-averaged approximations for inviscid flow models* (preprint).

Thank you for your attention